JOURNAL OF ECONOMICS TEACHING

Risk-Free Versus Risky Assets: Teaching a Portfolio Model with Application to the Stock Market

In this paper, we present an application where advanced undergraduate students can solve the expected utility portfolio model with a risk-free asset and a risky asset with both up and down returns in the stock market. With real stock market data, we use Excel Solver to find the portfolio decision and study how it changes when considering assets with different returns. Finally, we test students' portfolio decisions and their degree of risk aversion using different utility functions.

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1. Introduction

Imagine that you have an initial wealth of $10,000 that can be invested in a risk-free asset with an annual return of 3% or in a risky asset with two equally possible returns of 77% and -50% (e.g., the up and down returns in the stock market). *How much would you invest in the risky asset?*

To help advanced undergraduate students answer the above question, we present an application of the two-asset two-state portfolio model with a risk-free asset and a risky asset. Before introducing the model, we ask students how much they would invest in a risky asset taken from the stock market. Then, we introduce the standard portfolio model based on the expected utility theory and present a tutorial exercise that solves it using Excel Solver. Next, we carry out some comparative static analyses to facilitate students to understand the role of each parameter of the model. These exercises allow students to improve their knowledge of what happens to the amount of investment in the risky asset when, for example, either its return or the agent's initial wealth changes.

A central point in this paper is to link the model to Stock market data. To do so, we use the Yahoo finance website to obtain the returns of different assets with their respective probabilities in good and bad times (positive and negative returns, respectively). In particular, we perform an exercise where the individual decides how much of his initial wealth to allocate between the 10-year Treasury (the risk-free asset) and one risky asset among the following three possibilities: Facebook, Bitcoin, and S&P500. We also emphasize how relevant the degree of risk aversion is in the portfolio decision with the mentioned assets. Finally, we obtain the degree of risk aversion consistent with the average amount of investment in the risky asset obtained from students' answers.

Gravelle and Rees (2004) and Danthine and Donaldson (2014) are, for example, theoretical books in economics and financial economics, respectively, where the portfolio model is presented and analyzed. Theoretical textbooks do not cover information technology tools to solve the standard portfolio problem. However, Excel tools are increasingly being used for teaching economics, helping to improve learning outcomes (Barreto, 2015). Along these lines, many authors have used Excel Solver (see, for example, Berga, de Castro, & Silva, 2019; Silva & Xabadia, 2013; Benninga, 2010; Barreto, 2009) as a user-friendly and flexible tool for economic optimization problems (MacDonald, 1996).

An alternative approach for the expected utility portfolio model is the mean-variance analysis or modern portfolio theory that uses the variance of asset prices as a proxy for risk. Christou (2008) and Benninga (2014) present an interesting teaching application of this methodology to the stock market. Our approach is, therefore, complementary to them. Morone (2008) emphasizes the importance of testing the two theories using experimental data. He concludes that the expected utility theory better approximates agents' preferences than the mean-variance theory.

The proposed teaching material corresponds to an application related to the topic of uncertainty that has been implemented in the undergraduate course of Advanced Microeconomics that is compulsory for students earning an economics degree. This course contains advanced topics in uncertainty and asymmetric information, and its guidelines require special emphasis on applications for a better understanding of theory.

The organization of the paper is as follows. In Section 2, we introduce the intuition of the canonical portfolio model to familiarize students with its parameters. In Section 3, we formally present the model and state the effects of the parameters on the decision variable. In Section 4, we guide the student on how to build an Excel spreadsheet to solve the model and propose some comparative static analyses. In Section 5, we show students how to link the model's parameter with the stock market data and use it to compare the investment decisions with different risky-assets. Finally, we also use the model to analyze the effect of different degrees of risk aversion.

There are several ways to implement the material introduced in this paper depending on the instructor's idea and needs. Our suggestion is driven by yielding two main targets: motivating students and capturing their attention from scratch. To start, we recommend proposing an interactive exercise to students using real data in stock markets and testing students' portfolio decisions. This exercise can last twenty minutes (see Section 2). Then, the canonical two-asset two-state portfolio model in Section 3 can be presented, developed, and analytically solved in a two-hour master class. Third, the model is numerically solved in a two-hour computer class using Excel Solver, where the instructor guides the students with the step-by-step narrative accompanied with screenshots (Sections 4 and 5). We also recommend instructors teach these two sections using an overhead screen, especially if students are not familiar with the use of Excel Solver.

2. Risk-free vs. risky asset: testing students' portfolio decisions

This section introduces the intuition of the canonical portfolio model to familiarize students with the model and its parameters. The instructor will start the teaching session by presenting Table 1. Then, he will ask students how much would they invest in the risky asset.

Table 1: Risk-free versus risky asset

Initial wealth	10,000.00 \$ (w_0)	
Asset 1	Positive return	Negative return
Annual return	2.6% ($r = 0.026$)	-
Probability	1	-
Final wealth (if investing only in asset 1)	10,260.00 \$	-
Asset 2	Positive return	Negative return
Annual return	77.0% ($e_1 = 0.77$)	-49.5% ($e_2 = -0.495$)
Probability	0.5 (π)	0.5 (1- π)
Final wealth (if investing only in asset 2)	17,700 \$ (w_1)	5,050.00 \$ (w_2)

How much would you invest in asset 2 if you had an initial amount of \$10,000? (the rest of the initial wealth must be invested in asset 1) ________________________________

The instructor will point out that students have an initial wealth (w_0) of \$10,000 that can be invested in a risk-free asset with an annual return of 2.6% ($r=0.026$) or in a risky asset with two possible returns. The first expected return for the risky asset is 77% ($e_1=0.77$), corresponding to the annual return of an asset observed in the stock market (to be revealed in Section 5). In turn, the second expected return is -49.5% ($e_2=-0.495$) and corresponds to the average annual negative return observed for the same asset. Both returns for the risky asset have equal probability of occurring ($\pi=0.5$). The instructor will also point out what would be the initial and the final expected wealth if the student decided to invest only in either asset 1 or 2. In the first case, the final wealth would be \$10,260 with probability 1 while, in the second case, the final wealth would be equal to w_1=\$17,700 or w_2=\$5,050 with equal probability in both scenarios. The duration of the exercise will be around 20 minutes.

Once receiving the anonymous answers from students about their investment decision, the instructor will introduce the two-asset and two-state portfolio model by connecting the parameters of the model with their corresponding values in Table 1 (see Section 3).

3. The two-asset two-state portfolio model

This section presents the canonical two-asset two-state portfolio model with one risk-free asset that can be taught in two hours. The main goal is to decide the portfolio composition decision between a risk-free asset versus a risky asset. Following Table 1, consider an individual with an initial wealth $w_0 > 0$, who can invest in a risk-free asset with an annual return $r > 0$ or in a risky asset with two possible returns associated with two different states of the world (say, good and bad times). Formally, the return of the risky asset is either e_1 or e_2 with probabilities π and $1-\pi$, respectively. The return of the riskless asset lies between the two possible returns of the risky asset $e_1 > r > e_2$. The individual has a utility function $u(w) = \ln(w)$ strictly increasing and strictly concave in levels of wealth w , that is $u'(w) > 0$ and $u''(w) < 0$ (the individual is risk averse). Moreover, u satisfies the expected utility property.[1] The individual will decide the level of investment X in the risky asset that maximizes his expected utility.

The individual's portfolio decision problem can be stated mathematically as an optimization problem [P1] with a single variable X:

[P1]

$$max_{\{X\}}\, \pi u(w_1(X)) + (1-\pi)u(w_2(X))$$

$$w_1(X) = (w_0 - X)(1+r) + (1+e_1)X, \qquad (1)$$

$$w_2(X) = (w_0 - X)(1+r) + (1+e_2)X, \qquad (2)$$

$$0 \leq X \leq w_0. \qquad (3)$$

where $w_1(X)$ and $w_2(X)$ are the individual's contingent wealth level in each state of the world.[2]

To solve the optimization problem [P1], we obtain by replacing Equations (1) and (2) in the expected utility and obtaining the first-order condition of [P1]:

$$\frac{\pi(e_1-r)}{(1+r)w_0+(e_1-r)X} - \frac{(1-\pi)(r-e_2)}{(1+r)w_0+(e_2-r)X} = 0. \qquad (4)$$

The optimal level of investment can be obtained by isolating in Equation (4):

$$X = (1+r)w_0 \frac{\pi e_1+(1-\pi)e_2-r}{(r-e_2)(e_1-r)}\,. \qquad (5)$$

Let $\bar{e} = \pi e_1 + (1-\pi)\, e_2$ be the expected return of the risky asset. Note that by Equation (5), the amount of investment in the risky asset is zero if and only if $\bar{e} - r \leq 0$. Furthermore, the amount of investment in the risky asset will be w_0 if and only if $\bar{e} - r \geq \frac{(r-e_2)(e_1-r)}{(1+r)}$. The individual will invest some positive amount strictly smaller than his initial wealth in the risky asset if and only if the

[1]See, for example, Pindyck and Rubinfeld (2018) or Berga, de Castro, & Silva (2019) for a basic description of the expected utility theory, and Jehle and Reny (2011) and Mas-Colell, Whinston, and Green (1995) for an advanced and more extensive analysis.

[2]Writing $w_1(X) = w_0(1+r) + (e_1 - r)X$ and $w_2(X) = w_0(1+r) + (e_2 - r)X$ facilitates obtaining the first order condition.

following condition holds:

$$\frac{(r-e_2)(e_1-r)}{(1+r)} > \bar{e} - r > 0\,. \qquad (6)$$

Now we present some comparative statics using Equation (5) when Equation (6) holds. First, it is straightforward to see that an increase in either w_0 or e_2 implies an increase in X. Moreover, using partial derivative analysis, one can see that either a higher probability of being in good times π or a higher rate of return in good times e_1, generates an increase in X.[3] Therefore, by definition, an increase of only one of the three parameters in the expected return of the risky asset implies an increase for investment in that asset.[4] Finally, note the effect of r on X is negative when $e_2 > -1$ and it is not clear otherwise.[5]

4. Solving the portfolio model with Excel

The two-asset two-state portfolio model presented in the previous section can be solved numerically in a computer classroom exercise of one hour using Excel Solver. The instructor will guide the students with the step-by-step narrative and accompanying screenshots as explained below. We also recommend instructors perform the analysis using an overhead screen.

To do this classroom exercise, use Excel to solve the optimization problem [P1] where an agent with utility function u(w) =ln (w) has to decide the amount X to invest in the risky asset subject to $0 \leq X \leq w_0$. Then, implement the Generalized Reduced Gradient Nonlinear Optimization Method (GRG Nonlinear) available in Excel Solver.

Initial solution

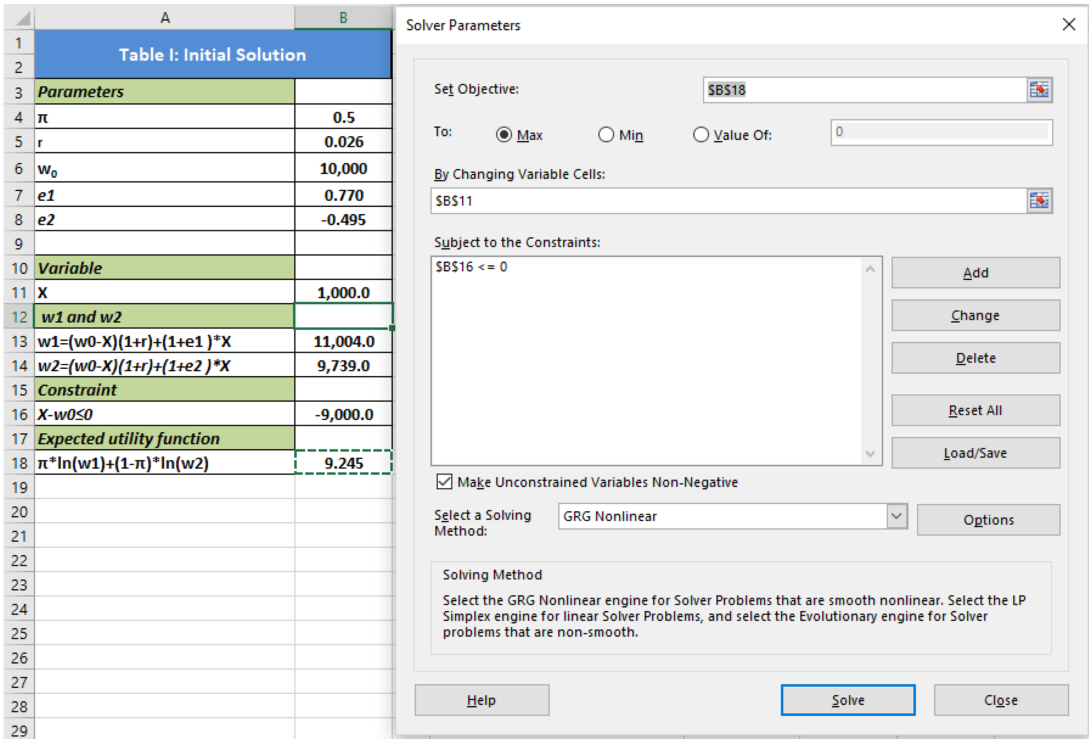
Starting with an Excel worksheet, build a table like Table I in Figure 1, setting up the initial optimization portfolio problem with the parameters, variable, constraint, and the expected utility function in column A. Rows 4 to 8 in column B include the parameter values from Table 1 in Section 2.

[3]To check them, note that the partial derivative of X with respect to π is positive: $\frac{\delta X}{\delta \pi} = (1+r)w_0 \frac{e_1 - e_2}{(r-e_2)(e_1-r)} > 0.$ Moreover, the partial derivative of X with respect to e_1 is also positive: $\frac{\delta X}{\delta e_1} = (1+r)w_0 \frac{\pi(e_1-r)-(\pi e_1+(1-\pi)e_2-r)}{(r-e_2)(e_1-r)^2} = (1+r)w_0 \frac{(1-\pi)(r-e_2)}{(r-e_2)(e_1-r)^2} > 0.$

[4]Notice that an increase in ē does not always imply an increase in X. We will give an example in Section 5b.

[5]To verify this, check that the sign of the partial derivative of the logarithm of X in Equation (5) with respect to r is $\frac{\delta \log(X)}{\delta r} = \frac{-(1+e_2)}{(1+r)(r-e_2)} + \frac{(\bar{e}-e_1)}{(e_1-r)(\bar{e}-r)} < 0.$

Figure 1: Setting up the problem for the initial solution



Next, introduce the initial value for the decision variable *X* in cell B11 (we set it at 1,000.0 but we recommend verifying the solution using other initial values to corroborate the optimal solution). The expressions for w_1 and w_2 in Equations (1) and (2) are introduced in cells B13 and B14 and the expression X-w_0 in cell B16 to later check constraint in Equation (3). Finally, the formula of the expected logarithm utility function is included in cell B18. Now choose Solver from the Data menu in Excel.[6] The *Solver Parameters* window will open. Set B18 as the *Objective Cell*, select *Max*, and set B11 as the *Changing Variable Cell* to add the decision variable . To introduce the constraint in cell B16, go to the *Subject to Constraints* box and select Add. The *Add Constraint* window will appear. In this window, tell Solver that cell B16 is *"≤ 0".* Then select *OK* since there are no more constraints to add. You will return to the *Solver Parameters* window as shown in Figure 1, click the box *Make Unconstrained Variables Non-Negative* (X ≥ 0), and Select GRG Nonlinear as the *Solving Method*.

Once defined all the necessary components of the model, click *Solve* in the *Solver Parameters* window. A window will appear telling you that Solver has found a solution. Select *Keep Solver Solution* and click *OK*. The portfolio decision solution appears in Table I of Figure 2. As you can see, the individual has maximized his expected utility by investing $2,951.3 in the risky asset *X* (cell B11) and the rest in the risk-free asset (w_0- X =$7,048.7 in cell B16). The individual's decision implies that the final wealth will be w_1 = $12,455.8 in good times (cell B13) or w_2 = $8,722.4 in bad times (cell B14). Note that the latter implies a wealth loss of w_2 - w_0 = $-1,277.6.

[6]If the command Solver does not appear in the Data menu, you can follow the instructions that appear in Excel help and type "Load the Solver Add-in".

Figure 2: Finding the initial solution

	A	B
1–2	**Table I: Initial Solution**	
3	***Parameters***	
4	π	0.5
5	r	0.026
6	w_0	10,000
7	*e1*	0.770
8	*e2*	-0.495
9		
10	***Variable***	
11	X	2,951.3
12	***w1 and w2***	
13	w1=(w0-X)(1+r)+(1+e1)*X	12,455.8
14	*w2=(w0-X)(1+r)+(1+e2)*X*	8,722.4
15	***Constraint***	
16	*X-w0≤0*	-7,048.7
17	***Expected utility function***	
18	π*ln(w1)+(1-π)*ln(w2)	9.252

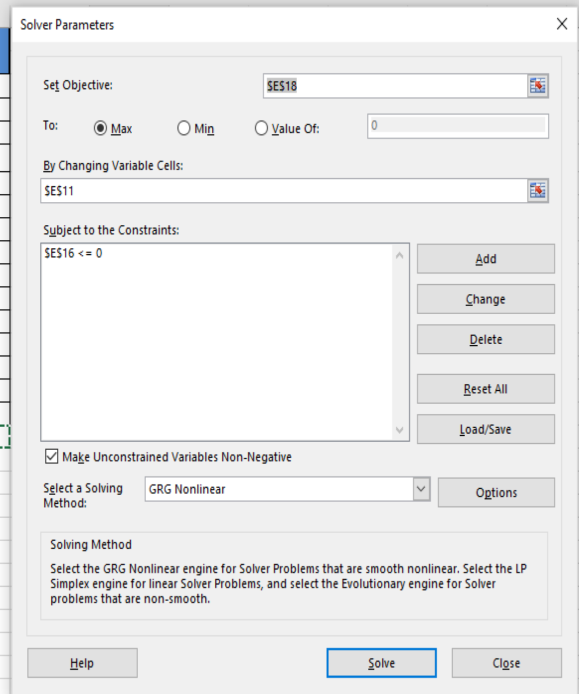
Comparative static analyses

Now, let us carry out some comparative static analyses by modifying the parameters of the model. To do that, duplicate all the components from Table I to a new Table II in the same worksheet as shown in Figure 3. To change the optimization problem open Solver and change the *Objective Cell* from B18 to E18 as well as the *Changing Variable Cell* from B11 to E11. Then, go to the *Subject to Constraints* box, select the constraints and click *Change*. The *Change Constraint* window will appear. In this window, replace B16 with E16. Then select *OK* and you will return to Excel.

The setting is ready for the comparative static analyses below, where the new values of the parameters in each case will be included in the corresponding cells of Table II, and then compare the new solution with the initial one in Table I. Before performing each exercise, reset the original values to the initial solution in Figure 2.

Figure 3: Setting up comparative static analyses

	A	B	C	D	E
1	Table I: Initial Solution			Table II: Comparative Analysis	
2					
3	*Parameters*			*Parameters*	
4	π	0.5		π	0.5
5	r	0.026		r	0.026
6	w_0	10,000		w_0	10,000
7	*e1*	0.770		*e1*	0.770
8	*e2*	-0.495		*e2*	-0.495
9					
10	*Variable*			*Variable*	
11	X	2,951.3		X	2,951.3
12	*w1 and w2*			*w1 and w2*	
13	w1=(w0-X)(1+r)+(1+e1)*X	12,455.8		w1=(w0-X)(1+r)+(1+e1)*X	12,455.8
14	*w2=(w0-X)(1+r)+(1+e2)*X*	8,722.4		*w2=(w0-X)(1+r)+(1+e2)*X*	8,722.4
15	*Constraint*			*Constraint*	
16	*X-w0≤0*	-7,048.7		*X-w0≤0*	-7,048.7
17	*Expected utility function*			*Expected utility function*	
18	π*ln(w1)+(1-π)*ln(w2)	9.252		π*ln(w1)+(1-π)*ln(w2)	9.252



<u>Comparative analysis 1</u>

What happens if the initial wealth increases from $10,000 to $20,000? In Section 3, we proved that when Equation (6) holds, the amount of investment in cell E11 increases with wealth in cell E6. To check this, it is necessary to change only the value of cell E6 from 10,000 to 20,000. Using Solver again, click *Solve* in the *Solver Parameters* window. See Figure 4 below. In this case, the amount of investment *X* has also doubled from $2,951.3 to $5,902.6 as can be formally checked in Equation (5). See cells B11 and E11 in Figure 4.

Figure 4: The effect of increasing the initial wealth w_o

	A	B	C	D	E
1	Table I: Initial Solution			Table II: Comparative Analysis	
2					
3	*Parameters*			*Parameters*	
4	π	0.5		π	0.5
5	r	0.026		r	0.026
6	w_0	10,000		w_0	20,000
7	*e1*	0.770		*e1*	0.770
8	*e2*	-0.495		*e2*	-0.495
9					
10	*Variable*			*Variable*	
11	X	2,951.3		X	5,902.6
12	*w1 and w2*			*w1 and w2*	
13	w1=(w0-X)(1+r)+(1+e1)*X	12,455.8		w1=(w0-X)(1+r)+(1+e1)*X	24,911.5
14	*w2=(w0-X)(1+r)+(1+e2)*X*	8,722.4		*w2=(w0-X)(1+r)+(1+e2)*X*	17,444.8
15	*Constraint*			*Constraint*	
16	*X-w0≤0*	-7,048.7		*X-w0≤0*	-14,097.4
17	*Expected utility function*			*Expected utility function*	
18	π*ln(w1)+(1-π)*ln(w2)	9.252		π*ln(w1)+(1-π)*ln(w2)	9.945

Comparative analysis 2:

What happens if only one of the three parameters defining the expected return of the risky asset increases? In Section 3, we also show that when only one of the parameters defining the expected return of the risky asset increases, the amount of the investment in that asset goes up. To check this, we next modify the parameters one by one keeping unchanged the other parameters as in the initial solution in Figure 2.

First, change the probability of being in good times from 0.5 to 0.6. Similar to the previous case, it is only necessary to write 0.6 as the value of cell E4 in Figure 3. Then, using Solver again and click *Solve* in the *Solver Parameters* window. In line with the theoretical comparative static analysis in Section 3, the amount of investment *X* in cell E11 increases from $2,951.3 to $6,299.6 (see cells B11 and E11 in Figure 5).

Figure 5: The effect of increasing the probability of being in good times

	A	B	C	D	E
1–2	Table I: Initial Solution			Table II: Comparative Analysis	
3	*Parameters*			*Parameters*	
4	π	0.5		π	0.6
5	r	0.026		r	0.026
6	w_0	10,000		w_0	10,000
7	*e1*	0.770		*e1*	0.770
8	*e2*	-0.495		*e2*	-0.495
9					
10	*Variable*			*Variable*	
11	X	2,951.3		X	6,299.6
12	*w1 and w2*			*w1 and w2*	
13	w1=(w0-X)(1+r)+(1+e1)*X	12,455.8		w1=(w0-X)(1+r)+(1+e1)*X	14,946.9
14	*w2=(w0-X)(1+r)+(1+e2)*X*	8,722.4		*w2=(w0-X)(1+r)+(1+e2)*X*	6,977.9
15	*Constraint*			*Constraint*	
16	*X-w0≤0*	-7,048.7		*X-w0≤0*	-3,700.4
17	*Expected utility function*			*Expected utility function*	
18	π*ln(w1)+(1-π)*ln(w2)	9.252		π*ln(w1)+(1-π)*ln(w2)	9.308

Second, change the return of the risky asset in good times from 0.77 to 0.87 by only changing the value of cell E7 in Figure 3. Then, using Solver again and click *Solve* in the *Solver Parameters* window. In this case, the amount of investment *X* increases from $2,951.3 to $3,768.3 (see cells B11 and E11 in Figure 6).

Figure 6: The effect of increasing the return of the risky asset in good times

	A	B	C	D	E
1	**Table I: Initial Solution**			**Table II: Comparative Analysis**	
2					
3	***Parameters***			***Parameters***	
4	**π**	**0.5**		**π**	**0.5**
5	**r**	**0.026**		**r**	**0.026**
6	w_0	**10,000**		w_0	**10,000**
7	***e1***	**0.770**		***e1***	**0.870**
8	***e2***	**-0.495**		***e2***	**-0.495**
9					
10	***Variable***			***Variable***	
11	**X**	**2,951.3**		**X**	**3,768.3**
12	***w1 and w2***			***w1 and w2***	
13	**w1=(w0-X)(1+r)+(1+e1)*X**	**12,455.8**		**w1=(w0-X)(1+r)+(1+e1)*X**	**13,440.4**
14	***w2=(w0-X)(1+r)+(1+e2)*X***	**8,722.4**		***w2=(w0-X)(1+r)+(1+e2)*X***	**8,296.7**
15	***Constraint***			***Constraint***	
16	***X-w0≤0***	**-7,048.7**		***X-w0≤0***	**-6,231.7**
17	***Expected utility function***			***Expected utility function***	
18	**π*ln(w1)+(1-π)*ln(w2)**	**9.252**		**π*ln(w1)+(1-π)*ln(w2)**	**9.265**

Finally, change the return of the risky asset in bad times from -0.495 to -0.395 by only changing the value of cell E8 in Figure 3. Then, using Solver again, click *Solve* in the *Solver Parameters* window. In this case, the amount of investment *X* increases from $2,951.3 to $5,290.1 (see cells B11 and E11 in Figure 7).

Figure 7: The effect of increasing the return of the risky asset in bad times

	A	B	C	D	E
1	**Table I: Initial Solution**			**Table II: Comparative Analysis**	
2					
3	***Parameters***			***Parameters***	
4	**π**	**0.5**		**π**	**0.5**
5	**r**	**0.026**		**r**	**0.026**
6	w_0	**10,000**		w_0	**10,000**
7	***e1***	**0.770**		***e1***	**0.770**
8	***e2***	**-0.495**		***e2***	**-0.395**
9					
10	***Variable***			***Variable***	
11	**X**	**2,951.3**		**X**	**5,290.1**
12	***w1 and w2***			***w1 and w2***	
13	**w1=(w0-X)(1+r)+(1+e1)*X**	**12,455.8**		**w1=(w0-X)(1+r)+(1+e1)*X**	**14,195.8**
14	***w2=(w0-X)(1+r)+(1+e2)*X***	**8,722.4**		***w2=(w0-X)(1+r)+(1+e2)*X***	**8,032.9**
15	***Constraint***			***Constraint***	
16	***X-w0≤0***	**-7,048.7**		***X-w0≤0***	**-4,709.9**
17	***Expected utility function***			***Expected utility function***	
18	**π*ln(w1)+(1-π)*ln(w2)**	**9.252**		**π*ln(w1)+(1-π)*ln(w2)**	**9.276**

Comparative analysis 3:

What happens if the return of the risk-free asset increases from 0.026 to 0.050? In Section 3, we saw that the effect of *r* on *X* is negative when the return of the risky asset in bad times is higher than -1. Note that the latter inequality holds in our initial solution. Now, change the return of the risk-free asset *r* from 0.026 to 0.05 by changing the value of cell E5. Using Solver again, click *Solve* in the *Solver Parameters* window. In this case, the amount of investment *X* in the risky asset decreases from $2,951.3 to $2,341.4 (see cells B11 and E11 in Figure 8).

Figure 8: The effect of increasing the return of the risk-free asset

	A	B	C	D	E
1–2	Table I: Initial Solution			Table II: Comparative Analysis	
3	*Parameters*			*Parameters*	
4	π	0.5		π	0.5
5	r	0.026		r	0.050
6	w_0	10,000		w_0	10,000
7	*e1*	0.770		*e1*	0.770
8	*e2*	-0.495		*e2*	-0.495
9					
10	*Variable*			*Variable*	
11	X	2,951.3		X	2,341.4
12	*w1 and w2*			*w1 and w2*	
13	w1=(w0-X)(1+r)+(1+e1)*X	12,455.8		w1=(w0-X)(1+r)+(1+e1)*X	12,185.8
14	*w2=(w0-X)(1+r)+(1+e2)*X*	8,722.4		*w2=(w0-X)(1+r)+(1+e2)*X*	9,224.0
15	*Constraint*			*Constraint*	
16	*X-w0≤0*	-7,048.7		*X-w0≤0*	-7,658.6
17	*Expected utility function*			*Expected utility function*	
18	π*ln(w1)+(1-π)*ln(w2)	9.252		π*ln(w1)+(1-π)*ln(w2)	9.269

5. Risk-free vs. risky asset: An application to the stock market

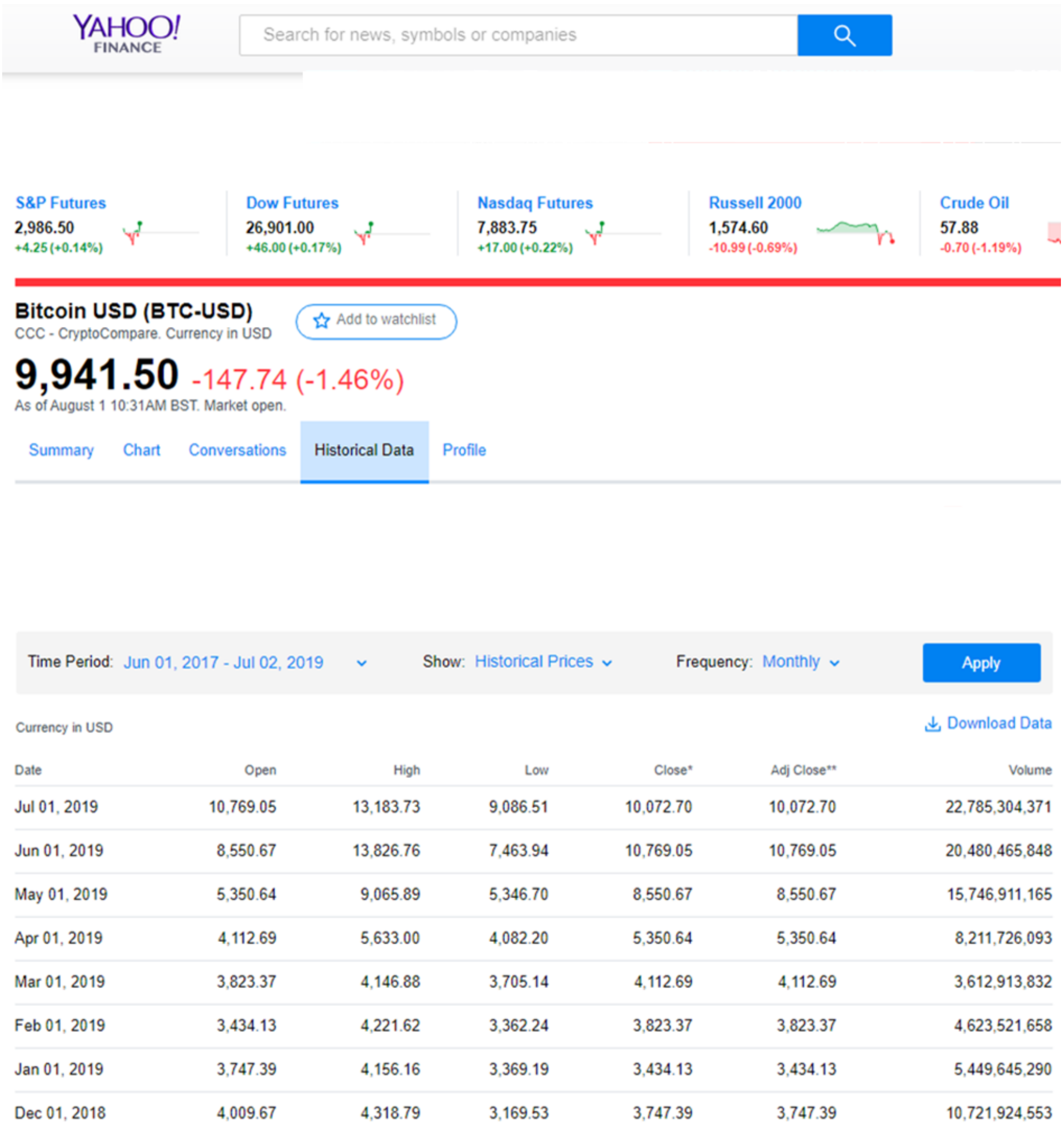
In this section the students can solve the portfolio model using data from the stock market in a teaching session of one hour.

Initial solution

Parametrize the initial simulation with the parameters that correspond to the ones displayed in Table 1: the risk-free *Asset 1* is the 10-year Treasury (US10Y) average yearly rate (2.6%) between June 2018 and July 2019, while the risky *Asset 2* corresponds to the average positive and negative returns observed in Bitcoin during the same period (77.0% and -49.5%, respectively). These parameters also correspond to the benchmark scenario when solving the portfolio model with Excel in Section 4 (see Figure 2).

To obtain these data, go to the Yahoo finance website https://finance.yahoo.com/ and write *Bitcoin* in the search window. Next, click on *Historical data* and choose the period and the frequency to be considered. In this case, choose monthly frequency from June 01, 2017 to July 02, 2019 (see Figure 9).

Figure 9: Stock data for the risky asset (Bitcoin)



Date	Open	High	Low	Close*	Adj Close**	Volume
Jul 01, 2019	10,769.05	13,183.73	9,086.51	10,072.70	10,072.70	22,785,304,371
Jun 01, 2019	8,550.67	13,826.76	7,463.94	10,769.05	10,769.05	20,480,465,848
May 01, 2019	5,350.64	9,065.89	5,346.70	8,550.67	8,550.67	15,746,911,165
Apr 01, 2019	4,112.69	5,633.00	4,082.20	5,350.64	5,350.64	8,211,726,093
Mar 01, 2019	3,823.37	4,146.88	3,705.14	4,112.69	4,112.69	3,612,913,832
Feb 01, 2019	3,434.13	4,221.62	3,362.24	3,823.37	3,823.37	4,623,521,658
Jan 01, 2019	3,747.39	4,156.16	3,369.19	3,434.13	3,434.13	5,449,645,290
Dec 01, 2018	4,009.67	4,318.79	3,169.53	3,747.39	3,747.39	10,721,924,553

Next, download the data to Excel and use the adjusting closed price after posting dividend (*Adj Close*). After downloading the data, calculate the annual return using a standard formula of the percentage change. For example, Figure 10 shows that the annual return of Bitcoin between July 01, 2018 and July 01, 2019 is ((B3-B15)/B15)*100=30.2. This means that if an individual bought Bitcoin on July 01, 2018 and sold it one year later, the return of the asset was equal to 30.2% (cell C3).

Figure 10: Finding the model's parameter using stock market data

	A	B	C	D	E	F	G
1		Risky Asset (Bitcoin)				Free-Risk Asset (US10Y)	Parameters of the model
2	Date	Adj Close**	Annual Return	Positive return	Negative return	Annual Return	
3	Jul 01, 2019	10,072.7	30.2	30.2		2.0	
4	Jun 01, 2019	10,769.1	68.7	68.7		2.0	Risky Asset (Bitcoin)
5	May 01, 2019	8,550.7	14.0	14.0		2.1	Average positive return (e_1)
6	Apr 01, 2019	5,350.6	-42.1		-42.1	2.5	77.0
7	Mar 01, 2019	4,112.7	-40.8		-40.8	2.4	Average negative return (e_2)
8	Feb 01, 2019	3,823.4	-63.0		-63.0	2.7	-49.5
9	Jan 01, 2019	3,434.1	-66.4		-66.4	2.6	Frecuency of positive return
10	Dec 01, 2018	3,747.4	-72.9		-72.9	2.7	7.0
11	Nov 01, 2018	4,009.7	-59.7		-59.7	3.0	Frecuency of negative return
12	Oct 01, 2018	6,342.6	-1.7		-1.7	3.2	7.0
13	Sep 01, 2018	6,623.7	51.9	51.9		3.1	Probability of positive return (π)
14	Aug 01, 2018	7,027.0	48.4	48.4		2.9	0.5
15	Jul 01, 2018	7,735.3	168.3	168.3		3.0	Probability of negative return ($1-\pi$)
16	Jun 01, 2018	6,385.4	157.4	157.4		2.8	0.5
17	May 01, 2018	7,502.2					
18	Apr 01, 2018	9,248.5					Free-Risk Asset (US10Y)
19	Mar 01, 2018	6,943.8					Average positive return (r)
20	Feb 01, 2018	10,334.4					2.6
21	Jan 01, 2018	10,226.9					
22	Dec 01, 2017	13,850.4					
23	Nov 01, 2017	9,946.8					
24	Oct 01, 2017	6,451.2					
25	Sep 01, 2017	4,360.6					
26	Aug 01, 2017	4,735.1					
27	Jul 01, 2017	2,883.3					
28	Jun 01, 2017	2,480.6					

Notice that the annual return of Bitcoin is negative between October 2018 and April 2019 (column E), while it is positive during the remaining seven months of the period considered (column D). Thus, the asset displayed a negative return during half of the period with an average value of -49.5% (cell G8), while the average positive return was equal to 77.0% (cell G6).

To get the parameters of the risk-free asset, go back to the Yahoo finance web page and write *Treasury Yield 10-Year* (US10Y) in the search window. Then, download the data and paste it into the previous Excel with Bitcoin (see Column F in Figure 10). Next, calculate the average return of the US10Y asset using the *Adj Close* price. In this case, Adj Close corresponds to the annual return of the asset. For example, the annual return of a US10Y bought on July 01, 2019 was equal to 2.0% (cell F3 in Figure 10). Now, with all this information, the student can obtain the parameters of the portfolio model:

- The positive return of the risky asset (cell G6) is equal to $e_1 = \frac{\text{AVERAGE(D3:D16)}}{100} = 0.770.$
- The negative return of the risky asset (cell G8) is equal to $e_2 = \frac{\text{AVERAGE(E3:E16)}}{100} = -0.495.$
- The probability of occurring (cell G14) is equal to $\pi = \frac{Months\ with\ positive\ return}{Total\ months} = \frac{7}{14} = 0.5.$
- The return of the risk-free asset (cell G20) is $r = \frac{AVERAGE(F3:F16)}{100} = 0.026.$
- The last parameter of the model is the initial wealth which has already been set at $W_0 =$ 10,000.

<u>Using the model to compare different risky assets</u>

Given a risk-free asset, the canonical portfolio model can be also used to select among several risky assets by choosing the one that generates the highest utility level.[7]

For example, imagine that an individual has an initial wealth of $10,000 and has to decide to invest this amount of money between the risk-free asset US10Y and one risky asset among the following three possibilities: Facebook, Bitcoin, and S&P500.

Table 2 shows the returns *e* and probabilities π of these assets between July 01, 2018 and July 01, 2019.[8] S&P500 displays a positive return of 9% and a negative return of 5% with a probability of 0.86 and 0.14, respectively. In turn, Facebook generates a positive return of 10% with a probability equal to 0.43 or a negative return of -18% with a probability of 0.57. Finally, the third option is Bitcoin that shows positive and negative return of 77% and -49.5%, respectively, with an equal probability (0.5). Now, the student only needs to solve the model by changing the parameters e_1, e_2, π considering each risky asset and comparing the levels of expected utility in each simulation. As shown in Table 2, the individual decides to invest all his initial wealth ($10,000) in S&P500 since this generates the highest expected level of utility (9.277). Notice that this decision takes place even though Bitcoin shows the highest expected return $\bar{e}_{Bitcoin} = 0.77 * 0.5+0.5 * (-0.495)=0.1375$. This result is due to the relatively high level of risk aversion that the individual has compared to individuals with other utility functions, as the ones analyzed in the next subsection.

Table 2: Comparing risky assets

Risky Asset	e_1	e_2	π	1-π	*X*	*w₁*	*w₂*	*Utility*	*Average return*
S&P500	0.09	-0.05	0.86	0.14	10,000	10,900	9,500	9.277	0.0704
Facebook	0.1	-0.18	0.43	0.57	0	10,260	10,260	9.236	-0.0596
Bitcoin	0.77	-0.495	0.5	0.5	2,951.3	12,456	8,722	9.252	0.1375

Finally, observe that the individual decides to invest zero in Facebook and $10,000 in S&P500. This aligns with the analysis presented in Section 3 where the difference between the expected return of the risky asset and the return of the risk-free asset $\bar{e} - r$ is crucial. In the case of Facebook $\bar{e} - r = -0.0596 - 0.026 \leq 0$, thus the amount of investment in the risky asset is zero. Moreover, in the case of S&P500, $X=w_0$ because $\bar{e} - r = 0.0704 - 0.026 = 0.0444$ is higher than $\frac{(r-e_2)(e_1-r)}{(1+r)} = \frac{(0.026+0.05)(0.09-0.026)}{(1+0.026)} = 0.0047$.

An interesting observation is that an increase in $\bar{e}$ does not always imply an increase in X. For this particular risk-averse agent an increase of the expected return from 0.0704 to 0.1375 implies a decrease in the amount invested in the risky asset from $10,000 to $2,951.3

The portfolio model with different degree of risk aversion

Until now, we have used a natural logarithmic utility function with a degree of risk aversion of 1 using the Arrow-Pratt coefficient of relative risk aversion.[9] It is also interesting to see how the portfolio decision changes when the individual becomes less risk averse.

[7]A more realistic but also more complex problem would be to diversify the investment among different risky assets but is not the objective of this paper.

[8]The positive and negative returns of each asset as well as their probabilities have been calculated following the same methodology as above in this section.

[9]The Arrow-Pratt coefficient of relative risk aversion is defined as $R(w) = -\frac{u''(w)}{u'(w)}w$. See Footnote 1 for references on expected utility theory.

In this section, we introduce the utility function $u(w) = w^{\alpha}$ where $(1\text{-}\alpha)$ is the Arrow-Pratt coefficient of relative risk aversion. Having α where 0<α< 1, means that the individual is risk averse, while α = 1 and α > 1 correspond to the case of risk neutrality and risk loving, respectively.[10] Figure 11 shows the simulated results of the portfolio model using this utility function and two different values of α: α=0.5 in Table I and α=0.6 in Table II, the latter representing a less risk-averse individual. To do this exercise, the student can use the same information in Figure 1 by changing the utility function (cell B18) and adding α in the parameters block (new cell B9). Then, duplicate all the components from Table I to a new Table II in the same worksheet by opening Solver and changing the cells from B to E as we did in the comparative static exercises in Section 4.

Remember that this scenario corresponds to a portfolio decision between the risky asset Bitcoin and the risk-free asset US10Y. Figure 11 shows that when α=0.5 in cell B9, the individual decides to invest *X=$5,902.6* in Bitcoin (cell B11) and the rest in US10Y ($4,097.8). However, when α=0.6 in cell E9, the agent becomes less risk averse and decides to invest *X=$7,321.4* of his initial wealth in the risky asset (see cell E11).

Figure 11: The effect of decreasing the degree of risk aversion

	A	B	C	D	E
1–2	Table I: Initial Solution			Table II: Comparative Analysis	
3	*Parameters*			*Parameters*	
4	π	0.5		π	0.5
5	r	0.026		r	0.026
6	w_0	10,000		w_0	10,000
7	*e1*	0.770		*e1*	0.770
8	*e2*	-0.495		*e2*	-0.495
9	α	0.5		α	0.6
10	*Variables*			*Variables*	
11	X	5,902.6		X	7,321.4
12	*w1 and w2*			*w1 and w2*	
13	w1=(w0-X)(1+r)+(1+e1)*X	14,651.5		w1=(w0-X)(1+r)+(1+e1)*X	15,707.1
14	*w2=(w0-X)(1+r)+(1+e2)*X*	7,184.8		*w2=(w0-X)(1+r)+(1+e2)*X*	6,445.5
15	*Constraints*			*Constraints*	
16	*X-w0≤0*	-4,097.4		*X-w0≤0*	-2,678.6
17	*Expected utility function*			*Expected utility function*	
18	π*w1^α+(1-π)*w2^α	102.903		π*w1^α+(1-π)*w2^α	261.175

Now, we can solve the canonical portfolio to select among several risky assets, and the given risk-free asset, but now considering utilities with different degrees of risk aversion *α*. Table 3 replicates Table 2 with a utility function u^{α} for *α=0.5* and *α=0.9*. In the first case, the individual decides to invest the total initial wealth in S&P500 while, in the second case, the individual prefers to invest all in Bitcoin. This last risky asset has a higher average return (13.75% vs. 7%) but also a higher probability of generating a negative return (0.5 vs. 0.14). Thus, the higher

[10]Berga, de Castro, and Silva (2019) define risk aversion for this utility with an interactive figure (see their Appendix). Moreover, it would be also interesting to consider other forms of utility functions like the one suggested by Holt and Laury (2002). By running an experiment the authors measure the degree of risk aversion and propose to use a hybrid utility function encompassing different risk behaviors like increasing relative risk aversion or constant relative risk aversion, among others.

the value of α (or equivalently, the smaller the degree of risk aversion) the more willing the individual is to invest in risky assets. Also note that for the natural logarithm utility the amount of investment in the risky asset was smaller (see Figure 2) since it is more risk averse.

Table 3: Comparing risky assets with different degree of risk aversion (1-α)

α=0.5									
Risky Asset	e_1	e_2	**Π**	***1-π***	***X***	w_1	w_2	***Utility***	***Average return***
S&P500	0.09	-0.05	0.86	0.14	10,000	10,900	9,500	103.4	0.0704
Facebook	0.1	-0.18	0.43	0.57	0	10,260	10,260	101.3	-0.0596
Bitcoin	0.77	-0.495	0.5	0.5	5,902	14,651	7,184	102.1	0.1375
α=0.9									
Risky Asset	e_1	e_2	**Π**	***1-π***	***X***	w_1	w_2	***Utility***	***Average return***
S&P500	0.09	-0.05	0.86	0.14	10,000	10,900	9,500	4,232.0	0.0704
Facebook	0.1	-0.18	0.43	0.57	0	10,260	10,300	4,074.4	-0.0596
Bitcoin	0.77	-0.495	0.5	0.5	10,000	17,700	5,050	4,404.0	0.1375

<u>Determining the degree of risk aversion of the students</u>

Remember that the teaching session of the portfolio model started in Section 2 by presenting Table 1 and asking students how much they would invest in asset 2 (Bitcoin) if they had an initial amount of $10,000. An interesting exercise consists of obtaining the degree of risk aversion consistent with the average amount of investment in the risky asset obtained from students' answers. To do this, we use Excel Solver to find the value of that generates the above average.

For example, we presented this exercise in a compulsory Advanced Microeconomics course at the University of Girona in Spain by using euros instead of dollars as the unit of measure. There were 62 anonymous responses (24 females and 38 males). The average response was 3,736 euros of investment in Asset 2 (Bitcoin). Figure 12 shows that the degree of risk aversion consistent with an amount X=3,736 euros is $(1-\alpha) = (1-0.2084)=0.7916$. Interestingly, we obtain small gender differences: males choose, on average, a higher amount of investment in the risky asset than females (3,895 euros vs. 3,485 euros). Thus, according to the portfolio model, males show a lower degree of risk aversion ($1-\alpha = 0.760$) than the one observed in females ($1-\alpha = 0.848$). The gender difference becomes more relevant when comparing the distribution of the amount of investment in the risky asset. Figures 13 and 14 show that the percentage of males investing more than 5,000 euros is higher than females' (21.1% vs. 8.4%, respectively). The instructor can ask students to do the same calculation in their class by adjusting the parameter alpha until finding the average amount of investment X in the risky asset.

Figure 12: The degree of risk aversion of the class

Table II: Comparative Analysis	
Parameters	
π	0.5
r	0.026
w_0	10,000
e1	0.770
e2	-0.495
α	0.2084
Variables	
X	3,735.9
w1 and w2	
w1=(w0-X)(1+r)+(1+e1)*X	13,039.5
*w2=(w0-X)(1+r)+(1+e2)*X*	8,313.6
Constraints	
X-w0≤0	-6,264.1
Expected utility function	
π*w1^α+(1-π)*w2^α	6.882

Figure 13: Females' distribution of investment in the risky asset

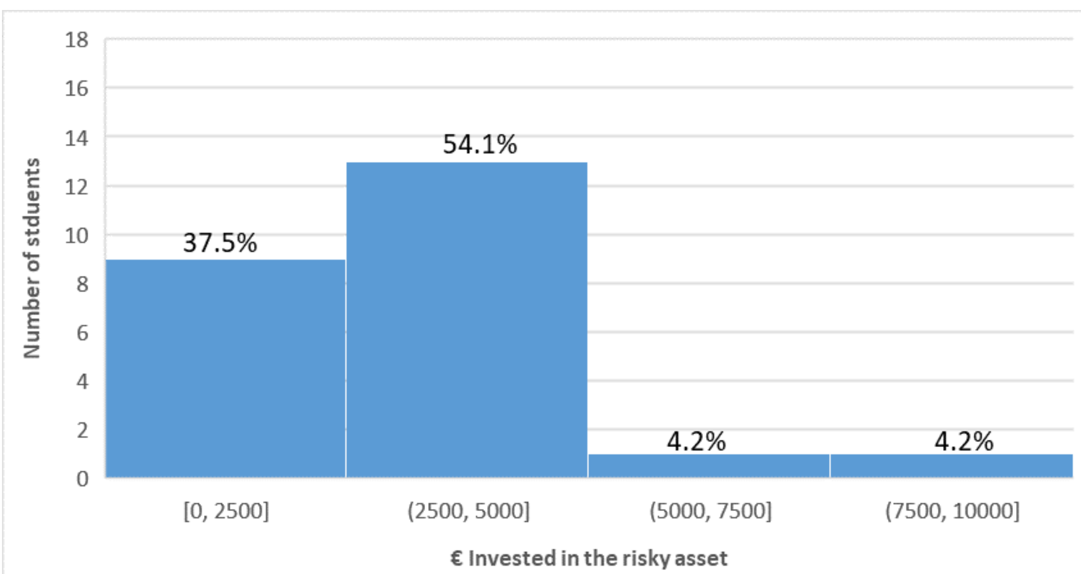
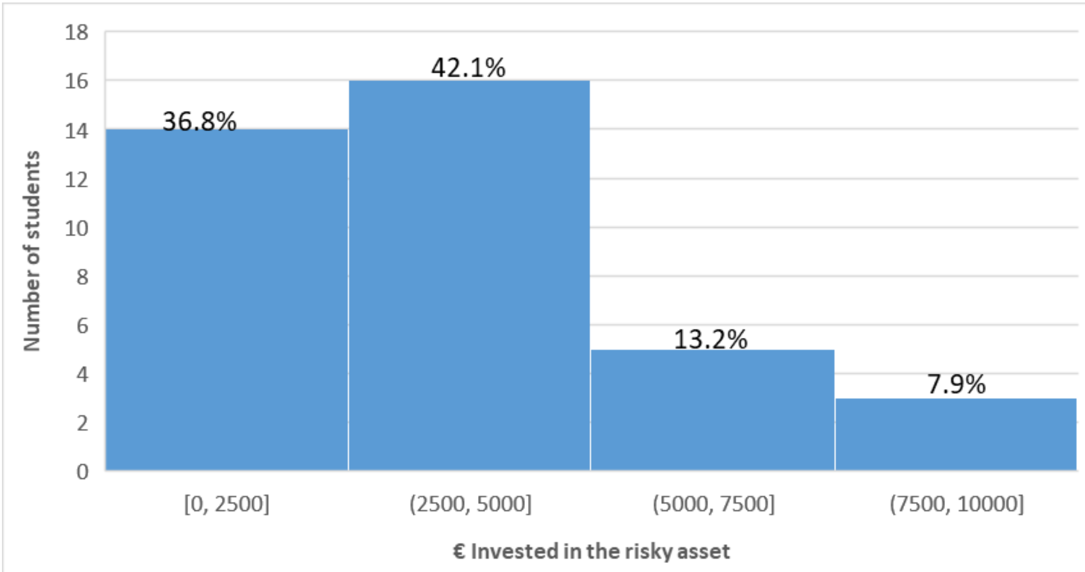


Figure 14: Males' distribution of investment in the risky asset



6. Concluding Remarks

In this paper we present an application where advanced undergraduate students can solve the expected utility portfolio model with a risk-free asset and a risky asset. First, we introduce the standard portfolio model and present a tutorial exercise that solves it by creating an Excel spreadsheet. Then, we do some comparative static analyses in order to help the students understand the role of each parameter of the model. These exercises allow students to improve their knowledge of what happens to the amount of investment in the risky asset when, for example, its return in good or bad times (or both) changes. We also analyzed the implications and intuitions for each exercise and get them back to the analytical solutions.

We link the model to the stock market data by comparing different assets. In particular, we introduce an exercise where an individual has an initial wealth of $10,000 and has to decide how to invest this amount of money between the risk-free asset US10Y and one risky asset among the following three possibilities: Facebook, Bitcoin, and S&P500. Finally, we show how the portfolio decision changes when the individual becomes less risk averse and test students' degrees of risk aversion by looking at their portfolio decisions.

Although we used a traditional classroom instruction approach, alternative instructional strategies, like flipped classroom, problem-based learning, etc., can also be applied to present the model and methodology in this document. However, this was not the objective of this work. Finally, an exercise that would be of interest for students consists of incorporating more than two states of nature for the risky asset which we do not consider in the present paper.

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